Lecture 5

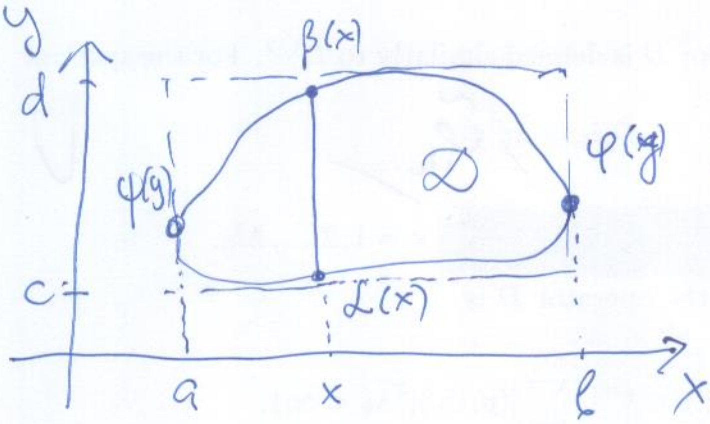
Methods of Integration

The main idea is to reduce the double integral to an iterated integral, i.e. a series (two) integrals of one variable, each being directly solvable.

Normal domains on $\mathbb{R}^2$

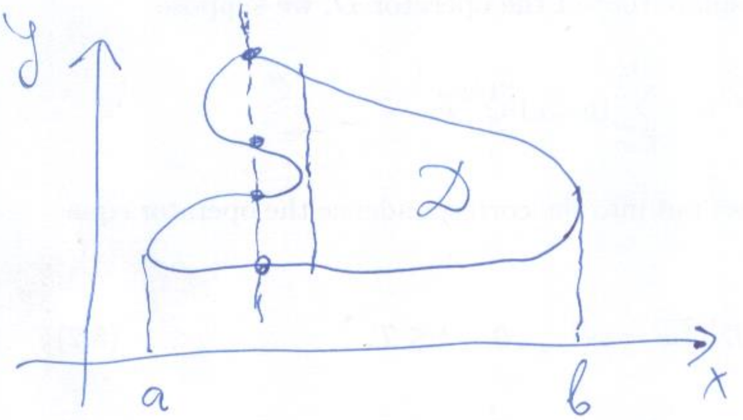
A domain D is called normal if

- the projection of D onto either the x-axis or the y-axis is bounded by the two values, a and b.
- any line perpendicular to this axis that passes between these two values intersects the domain in an interval whose ~~p~~ endpoints are given by the graphs of two functions α and β.



Looks similar for y-axis case.

Not normal domain



We can split $\mathcal{D}$ into a sum of normal domains $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$

Let's calculate the volume of the cylinder

$$V = \iint_{\mathcal{D}} f(x, y)\, dx\, dy.$$

Let take a plane $x = const$, for $a < x < b$.

We get a trapezoidal region $\tilde{S}(x)$ bounded from above by function $f(x,y)$ and base interval

$$\alpha(x) \le y \le \beta(x)$$

The area of this region is equal to

$$\tilde{S}(x) = \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy$$

Then the volume of the cylinder can be calculated as an integral of $\tilde{S}(x)$ on $[a, b]$

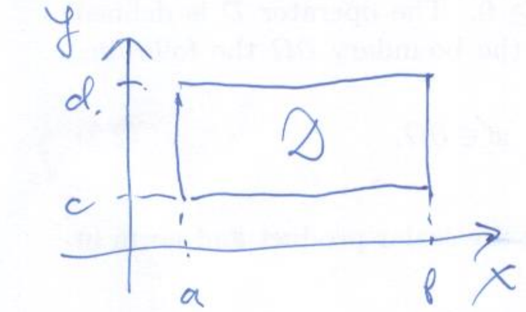
$$V = \int_a^b \tilde{S}(x)\,dx = \int_a^b \left(\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy \right) dx$$

$$= \int_a^b dx \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy \,\cancel{dx}$$

We can compute the same integral in the following way

$$V = \int_c^d dy \int_{\psi(y)}^{\varphi(y)} f(x,y)\, dx.$$

Example 1



$a = 1 \quad b = 4$

$c = 2 \quad d = 3$

$f(x,y) = xy$

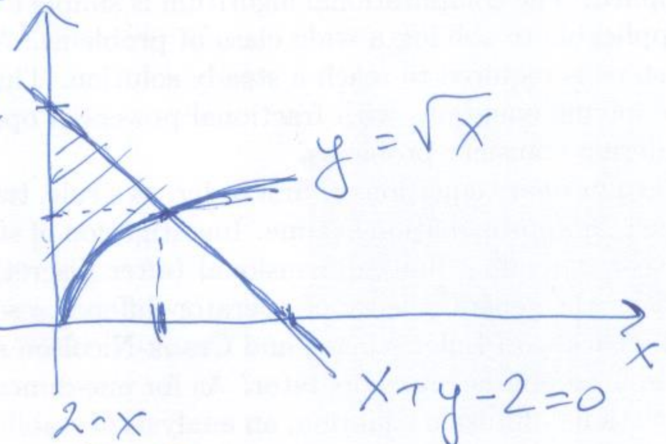
$$\iint_D xy\, dx\, dy = \int_1^4 dx \int_2^3 xy\, dy$$

$$= \int_1^4 x \frac{y^2}{2}\Big|_2^3 dx = \frac{5}{2}\int_1^4 x\, dx = \frac{5}{4}x^2\Big|_1^4.$$

Example 2 Change a double integral as a sequence of one-variable integrals (2 possibilities)

$$V = \iint_D f(x,y)\,dx$$

Region D is defined by Ox axis, parabola $y=\sqrt{x}$, and the line $x+y=2$



a) $V = \int_0^1 dx \int_{\sqrt{x}}^{2-x} f(x,y)\,dy$

b) $V = \int_0^1 dy \int_0^{y^2} f(x,y)\,dx + \int_1^2 dy \int_0^{2-y} f(x,y)\,dx$

In the second case it is convenient to consider two integrals.

Change of variables in Double integrals

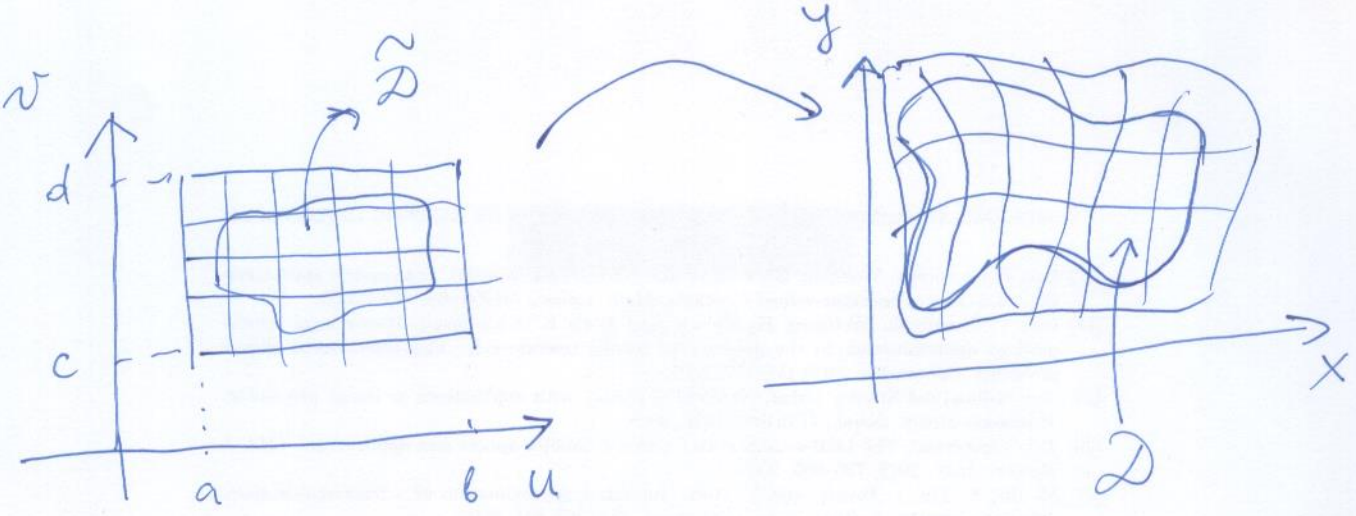
It is often advantageous to calculate $\iint_D f(x,y)\,dx\,dy$ in a coordinate system other than the xy-coordinates.

Lets suppose that

$$\begin{cases} x = \varphi(u,v) \\ y = \psi(u,v) \end{cases}$$

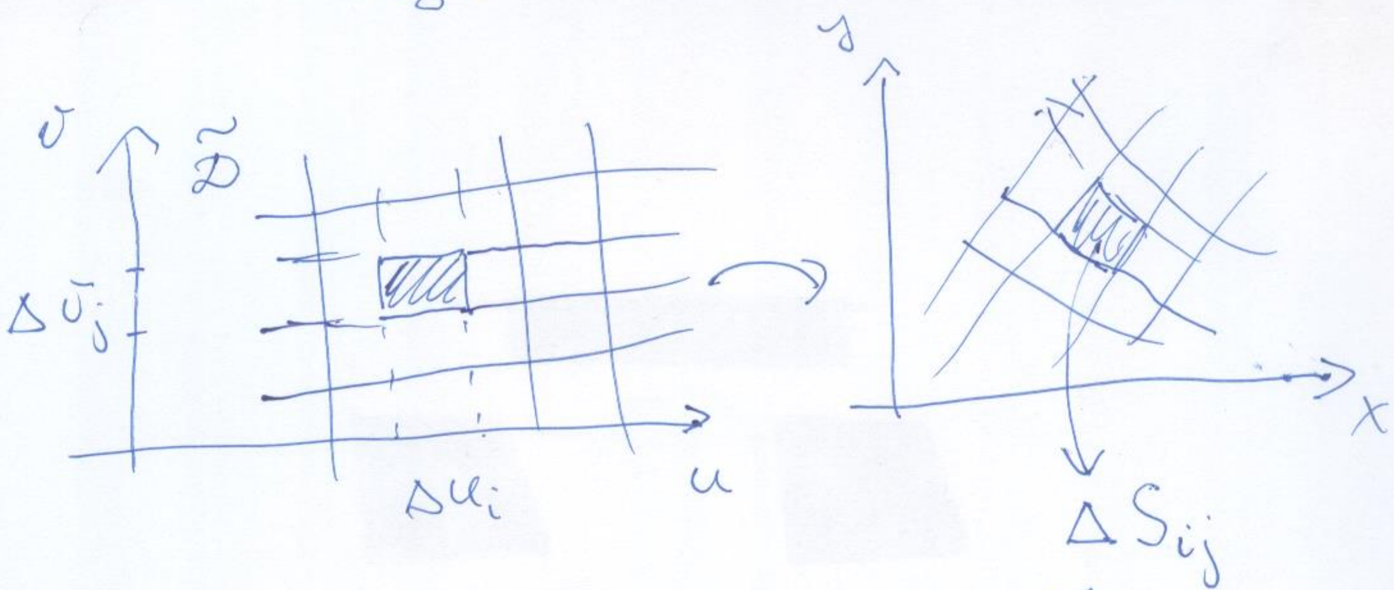
is a 1-1 map of a region $\tilde{D}$ in the uv-plane onto a region D in the xy-plane.

Let's suppose that $\tilde{D}$ is contained in a rectangle $[a,b]\times[c,d]$ on which both functions φ and ψ are differentiable.



A partition of $[a, b]$ into n subintervals of width Δu_i and $[c, d]$ into m subintervals of width Δv_j covers $\tilde{D}$ with a collection of rectangles.

Let's take a sufficiently small $h > 0$, then for $\boxed{\Delta u_i < h, \ \Delta v_i < h}$ the image of the rectangles in $\tilde{D}$ under mapping (φ, ψ) is a collection of regions covering D, which are approximately parallelograms



We calculate the area of the obtained parallelogram ΔS_{ij} and then a ~~total~~ area of region D is approximated as

$$S \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta S_{ij}$$

It is well known the area of the image of rectangle is equal to

$$\Delta S_{ij} \approx |J_{ij}| \, \Delta u_i \, \Delta v_j,$$

where J is determinant of Jacobi matrix

<u>Definition</u> The cross (vector) product of two vectors $\vec{a}$ and $\vec{b}$ is defined only in three-dimensional space and is denoted by

$$\vec{a} \times \vec{b}.$$

The cross product is defined as a vector $\vec{c}$ that is perpendicular (orthogonal) to both $\vec{a}$ and $\vec{b}$

$$\vec{a} \times \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \sin(\varphi) \, \vec{n}$$

where

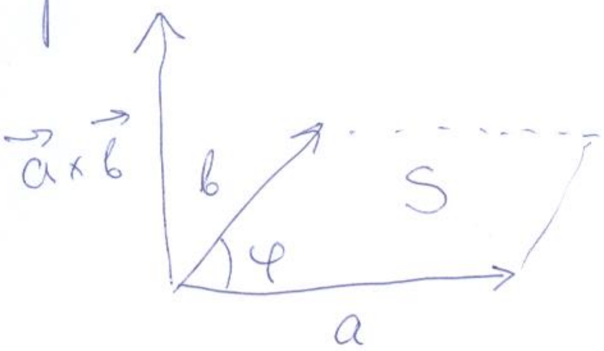
$\|\vec{a}\|, \|\vec{b}\|$ are the magnitudes of (length) of vectors $\vec{a}$ and $\vec{b}$

φ is the angle between $\vec{a}$ and $\vec{b}$.

$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$
$$\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The magnitude of the cross product can be interpreted as the positive area of the parallelogram



$\vec{a} = (a_1, a_2, 0)$

$\vec{b} = (b_1, b_2, 0)$

$$S = \| \mathcal{J} \|$$

$$= |a_1 b_2 - b_1 a_2|.$$

$$\mathcal{J} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix}$$

$$= \vec{k} \, (a_1 b_2 - b_1 a_2)$$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$$\vec{a} = \left(\frac{\partial \varphi}{\partial u} \Delta u, \frac{\partial \psi}{\partial u} \Delta u \right)$$

$$\vec{b} = \left(\frac{\partial \varphi}{\partial v} \Delta v, \frac{\partial \psi}{\partial v} \Delta v \right)$$

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} \frac{\partial \varphi}{\partial u} \Delta u & \frac{\partial \psi}{\partial u} \Delta u \\ \frac{\partial \varphi}{\partial v} \Delta v & \frac{\partial \psi}{\partial v} \Delta v \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \psi}{\partial u} \\ \frac{\partial \varphi}{\partial v} & \frac{\partial \psi}{\partial v} \end{vmatrix} \Delta u \, \Delta v$$

$$= \begin{vmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{vmatrix} \Delta u \, \Delta v$$

$$J = \begin{vmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{vmatrix}_{(u_i, v_i)}$$

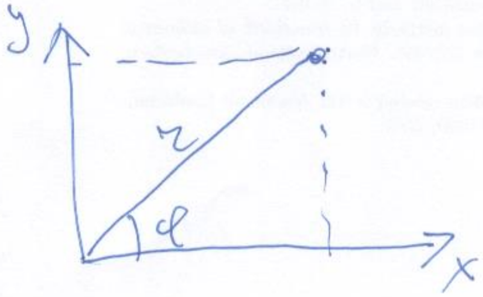
In the limit as $h \to 0$, the double sum

$$V \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(\tilde{x}_i, \tilde{y}_j) \Delta x_i \Delta y_j$$

leads to a double integral

$$V = \iint_{D} f(x,y)\,dx\,dy = \iint_{\tilde{D}} f(\varphi(u,v), \psi(u,v)) \times |J|\,du\,dv.$$

Polar coordinates



The polar coordinates

r - radius, a distant from a reference point (x, y)

φ - an angle from a reference direction.

The polar coordinates r and φ can be converted to the Cartesian coordinates by using the trigonometric functions

$$\boxed{\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}} \qquad \begin{aligned} &0 \le \varphi < 2\pi \\ &r > 0. \end{aligned}$$

The Jacobian of this transformation is calculated by

$$J = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\left(\sin^2\varphi + \cos^2\varphi\right)$$

$$= \underline{\underline{r}}.$$

$$\iint_{\mathcal{D}} f(x,y)\,dx\,dy = \iint_{\mathcal{D}} f(r\cos\varphi, r\sin\varphi)\, r\,dr\,d\varphi.$$

Example. Find the area of the region D defined by curves:

Equation of a circle

$$x^2+y^2=4x \Rightarrow (x-2)^2+y^2=4$$

radius $r=2$, the center $(2,0)$

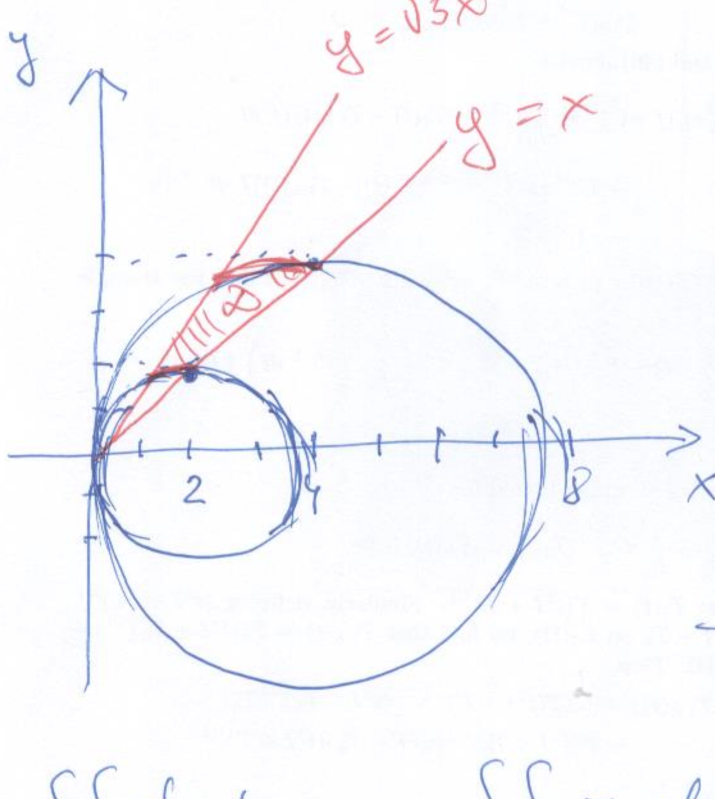
$$x^2+y^2=8x \Rightarrow (x-4)^2+y^2=16$$

$r=4$, center $(4,0)$

$$y=x$$

$$y=\sqrt{3}x$$

Equations of lines with slopes 1 and $\sqrt{3}$



$$x = r\cos\varphi$$

$$y = r\sin\varphi$$

$$\frac{\pi}{4} \le \varphi \le \frac{\pi}{3}$$

$$4\cos\varphi \le r \le 8\cos\varphi$$

Show it!

$$S = \iint_D dx\,dy = \iint_D r\,dr\,d\varphi$$

$$= \int_{\pi/4}^{\pi/3} d\varphi \int_{4\cos\varphi}^{8\cos\varphi} r\,dr = \ ?$$